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On the Boundedness of Solutions of Perturbed Linear Systems

TADAYUKI HARA, TOSHIAKI YONEYAMA, AND YOSHINORI OKAZAKI

Department of Mathematical Sciences, College of Engineering, University of Osaka Prefecture, Sakai, Osaka, 591 Japan

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1. INTRODUCTION

Consider the following systems of differential equations:

$$x' = A(t)\,x, \tag{L}$$

$$y' = A(t)\,y + g(t, y), \tag{PL}$$

$$y' = A(t)\,y + h(t), \tag{PL_h}$$

where x, y, g, h are n-vectors, $A(t)$ is a continuous $n \times n$ matrix for $t \geqslant 0$, $g(t, y)$ is continuous for $t \geqslant 0$, $y \in \mathbf{R}^n$, and $h(t)$ is continuous for $t \geqslant 0$.

Strauss and Yorke [8, 9] have studied the perturbing uniform asymptotically stable systems, and Furuno and Hara [4] have shown some more detailed results. Bernfeld [1] and Lovelady [6] have studied the perturbing uniformly bounded and uniformly ultimately bounded systems. On the other hand Coppel [2, 3] has studied the boundedness of solutions of (PL_h) from the point of view of the dichotomy theory. Lovelady [5] referred to the connections between the perturbation problem and the dichotomy theory.

Here we shall give some further results on the boundedness of solutions of perturbed linear systems.

This paper is much influenced by Strauss and Yorke [9].

The purpose of this paper is to prove theorems on the perturbation from (L) to (PL) and (PL_h) of uniform boundedness (Theorem 3.1), uniform boundedness and ultimate boundedness (Theorem 4.1), and uniform boundedness and uniform ultimate boundedness (Theorem 5.1).

Let G_0 be the class of functions $g(t, y)$ such that $\| g(t, y)\| \leqslant \gamma(t)\, \phi(\| y \|)$ for all $t \geqslant 0$ and $\| y \| \geqslant R$, where $\int_0^\infty \gamma(t)\, dt < \infty$ and $\phi(r)$ is a positive, continuous and nondecreasing function on $r \geqslant r_0 > 0$ and

$$\int_{r_0}^{\infty} \frac{dr}{\phi(r)} = \infty.$$

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We prove in Theorem 3.1 and Theorem 4.1 that if the solutions of (L) are uniformly bounded (hereafter called UB) and if $g(t, y) \in G_0$, then the solutions of (PL) are UB. If the solutions of (L) are UB and ultimately bounded (hereafter called UltB) and if $g(t, y) \in G_0$, then the solutions of (PL) are UB and ULtB. If $\int_0^\infty \|h(t)\| \, dt = \infty$, then there exists a matrix $A(t)$ such that the solutions of (L) are UB and UltB and the solutions of (PL_h) are not even UB. If in addition there exists a sequence $\{t_m\}$ such that $t_m \to \infty$ and $\|\int_0^{t_m} h(s) \, ds\| \to \infty$ as $m \to \infty$, then such a matrix $A(t)$ can be chosen to be bounded on $[0, \infty)$.

Let $\tilde{G}_0$ be the class of functions $g(t, y)$ such that $\|g(t, y)\| \leqslant \gamma(t)\|y\|^\beta$ $(0 \leqslant \beta < 1)$ for all $t \geqslant 0$ and $\|y\| \geqslant R$, where $\sup_{t \geqslant 0} e^{-t} \int_0^t e^s \gamma(s) \, ds < \infty$.

We prove in Theorem 5.1 that if the solutions of (L) are UB and uniformly ultimately bounded (hereafter called UUB) and if $g(t, y) \in \tilde{G}_0$, then the solutions of (PL) are UB and UUB. If there exists a sequence $\{t_m\}$ such that $t_m \to \infty$ and $e^{-t_m} \int_0^{t_m} e^s \|h(s)\| \, ds \to \infty$ as $m \to \infty$, then there exists a matrix $A(t)$ such that the solutions of (PL_h) are not even UB.

In Theorem 5.2, we prove that if $\sup_{t \geqslant 0} e^{-t} \int_0^t e^s \|A(s)\| \, ds < +\infty$ and the solutions of (L) are UB and UUB, then the solutions of (PL_h) are UB and UUB if and only if $\sup_{t \geqslant 0} \|e^{-t} \int_0^t e^s h(s) \, ds\| < +\infty$.

2. Definitions and Lemmas

Let $\mathbf{R}^n$ denote the Euclidean n-space. For $x \in \mathbf{R}^n$, let $\|x\| = \sum_{i=1}^n |x_i|$. For an $n \times n$ matrix $A = (a_{ij})$, define the norm $\|A\|$ of A by $\|A\| = \sum_{i,j=1}^n |a_{ij}|$. I denotes the $n \times n$ identity matrix.

We next present the definitions of boundedness of solutions. The definitions are stated for the system,

$$x' = f(t, x), \tag{2.1}$$

where $f(t, x)$ is continuous from $[0, \infty) \times \mathbf{R}^n$ to $\mathbf{R}^n$. We denote by $x(t, t_0, x_0)$ the solutions of (2.1) through (t_0, x_0).

Definition 2.1. The solutions of (2.1) are *uniformly bounded* (UB) if for any $\alpha > 0$, there exists $\beta(\alpha) > 0$ such that $\|x_0\| \leqslant \alpha$ implies that

$$\|x(t, t_0, x_0)\| < \beta(\alpha) \qquad \text{for all} \quad t_0 \geqslant 0 \qquad \text{and} \quad t \geqslant t_0 \geqslant 0.$$

Definition 2.2. The solutions of (2.1) are *ultimately bounded* (UltB) for bound B, if there exist $B > 0$ and $T > 0$ such that for every solution $x(t, t_0, x_0)$ of (2.1), $\|x(t, t_0, x_0)\| < B$ for all $t \geqslant t_0 + T$, where B is independent of the particular solutions while T may depend on each solution.

DEFINITION 2.3. The solutions of (2.1) are *uniformly ultimately bounded* (UUB) for bound B, if there exists $B>0$ and if corresponding to any $\alpha>0$, there exists $T(\alpha)>0$ such that $\|x_0\|<\alpha$ implies that

$$\|x(t,t_0,x_0)\|<B \qquad \text{for all} \quad t_0\geqslant 0 \qquad \text{and} \quad t\geqslant t_0+T(\alpha).$$

We now give a lemma which is concerned with uniform boundedness in terms of Liapunov functions.

Hereafter a Liapunov function $V(t,x)$ will be assumed to be a scalar continuous function which satisfies locally a Lipschitz condition with respect to x.

LEMMA 2.1. *Suppose that there exists a Liapunov function $V(t,x)$ defined on $0\leqslant t<\infty$, $\|x\|\geqslant R$, where R may be large, which satisfies the following conditions*:

(i) $a(\|x\|)\leqslant V(t,x)\leqslant b(\|x\|)$, *where $a(r)$, $b(r)$ are continuous and increasing and $a(r)\to\infty$ as $r\to\infty$,*

(ii) $$\dot V_{(2.1)}(t,x)\equiv\limsup_{h\to 0+}(1/h)\{V(t+h,x+hf(t,x))-V(t,x)\}$$
$$\leqslant \alpha(t)\,\phi(V(t,x)),$$

where $\alpha(t)\geqslant 0$ is a continuous function on $t\geqslant 0$ and $\phi(r)$ is a positive, continuous, and nondecreasing function on $r\geqslant r_0$ such that

$$\int_0^\infty \alpha(t)\,dt<\infty \qquad \text{and} \qquad \int_{r_0}^\infty \frac{du}{\phi(u)}=\infty.$$

Then the solutions of (2.1) *are UB.*

We shall sketch the proof of Lemma 2.1.

Let

$$U(t,x)=-\int_0^t \alpha(s)\,ds+\int_{r_0}^{V(t,x)}\frac{du}{\phi(u)}. \tag{2.2}$$

Then for $t\geqslant 0$ and $\|x\|\geqslant R$ we have

$$\dot U_{(2.1)}(t,x)\leqslant 0 \tag{2.3}$$

and

$$\int_{r_0}^{a(\|x\|)}\frac{du}{\phi(u)}-L\leqslant U(t,x)\leqslant\int_{r_0}^{b(\|x\|)}\frac{du}{\phi(u)}, \tag{2.4}$$

where $L = \int_0^\infty \alpha(t)\,dt$. Let

$$\tilde{a}(r) = \int_{r_0}^{a(r)} \frac{du}{\phi(u)} - L \qquad \text{and} \qquad \tilde{b}(r) = \int_{r_0}^{b(r)} \frac{du}{\phi(u)}.$$

Then $\tilde{a}(r)$ and $\tilde{b}(r)$ are continuous and increasing and $\tilde{a}(r) \to \infty$ as $r \to \infty$. Using the well-known theorem on uniform boundedness [10, Theorem 10.2], (2.3), and (2.4) show that the solutions of (2.1) are UB.

Next we give a lemma on uniform boundedness and uniform ultimate boundedness.

LEMMA 2.2. *Suppose that there exists a Liapunov function $V(t, x)$ defined on $0 \leqslant t < \infty$, $\|x\| \geqslant R$, where R may be large, which satisfies the following conditions*:

(i) $a(\|x\|) \leqslant V(t, x) \leqslant b(\|x\|)$, *where $a(r), b(r)$ are continuous and increasing and $a(r) \to \infty$ as $r \to \infty$*,

(ii) $\dot{V}_{(2.1)}(t, x) \leqslant -\{c - \alpha_1(t)\}\, V + \alpha_2(t)\, V^\beta \quad (0 \leqslant \beta < 1)$

where $c > 0$ is a constant and $\alpha_i(t) \geqslant 0$ $(i = 1, 2)$ are continuous functions satisfying

$$\limsup_{(t,v) \to (\infty,\infty)} \frac{1}{v} \int_t^{t+v} \alpha_1(s)\,ds < c \tag{2.5}$$

and

$$\sup_{t \geqslant 0} \int_t^{t+1} \alpha_2(s)\,ds < +\infty. \tag{2.6}$$

Then the solutions of (2.1) *are UB and UUB.*

We shall sketch the proof of Lemma 2.2.

Choose $\varepsilon > 0$ such that $\varepsilon < c/3$ and

$$\limsup_{(t, s-t) \to (\infty,\infty)} \frac{1}{s-t} \int_t^s \alpha_1(\tau)\,d\tau \leqslant c - 3\varepsilon.$$

Then for some positive constant M, we have

$$\exp\left\{-c(s-t) + \int_t^s \alpha_1(\tau)\,d\tau\right\} \leqslant M e^{-2\varepsilon(s-t)} \qquad \text{for} \quad s \geqslant t \geqslant 0.$$

Define $U(t, x)$ by

$$U(t, x) = \{V(t, x)\, E(t)\}^{1-\beta},$$

where $E(t) = e^{-\varepsilon t} \int_t^\infty e^{\varepsilon s} \exp\{-c(s-t) + \int_t^s \alpha_1(\tau)\, d\tau\}\, ds$. Then, for $t \geqslant 0$ and $\|x\| \geqslant R$, we have

$$\{e^{-c} a(\|x\|)\}^{1-\beta} \leqslant U(t, x) \leqslant \{(M/\varepsilon)\, b(\|x\|)\}^{1-\beta} \tag{2.7}$$

and

$$\dot{U}_{(2.1)} \leqslant -\sigma U + \tilde{M} \alpha_2(t),$$

where $\sigma = \varepsilon(1-\beta) > 0$ and $\tilde{M} = (1-\beta)(M/\varepsilon)^{1-\beta} > 0$. Thus we have

$$U(t, x(t)) \leqslant U(t_0, x_0)\, e^{-\sigma(t-t_0)} + \tilde{M} e^{-\sigma t} \int_{t_0}^{t} e^{\sigma s} \alpha_2(s)\, ds$$

for $t \geqslant t_0 \geqslant 0$, where $x(t)(x(t_0) = x_0)$ is any solution of (2.1). By (2.6) we have

$$\sup_{t \geqslant 0} e^{-\sigma t} \int_0^t e^{\sigma s} \alpha_2(s)\, ds < +\infty.$$

Then (2.7) implies that the solutions of (2.1) are UB and UUB, thus proving the lemma.

For the linear system (L),

(i) uniform boundedness and uniform stability are equivalent,

(ii) ultimate boundedness and asymptotic stability are equivalent,

(iii) uniform boundedness and uniform ultimate boundedness are equivalent to uniform asymptotic stability [10, Theorem 11.2].

Therefore, if $X(t)$ is a fundamental matrix for (L), the following lemma holds [2, p. 54, Theorem 1].

LEMMA 2.3. *The solutions of* (L) *are*

(i) *UB if and only if there exists a positive constant* $K \geqslant 1$ *such that*

$$\|X(t) X^{-1}(s)\| \leqslant K \qquad \text{for all} \quad t \geqslant s \geqslant 0,$$

(ii) *UltB if and only if*

$$\|X(t)\| \to 0 \qquad \text{as} \quad t \to \infty,$$

(iii) *UB and UUB if and only if there exist positive constants* $K \geqslant 1$ *and* $\lambda > 0$ *such that*

$$\|X(t) X^{-1}(s)\| \leqslant K e^{-\lambda(t-s)} \qquad \text{for all} \quad t \geqslant s \geqslant 0.$$

We consider the following classes of continuous matrices $A(t)$;

$$\mathscr{A}_B = \{A(t); \sup_{t\geqslant 0} \|A(t)\| < \infty\},$$

$$\mathscr{A}_A = \left\{A(t); \sup_{t\geqslant 0} \int_t^{t+1} \|A(s)\|\, ds < \infty\right\},$$

$$\mathscr{A}_E = \left\{A(t); \sup_{t\geqslant 0} \left\|e^{-t} \int_0^t e^s A(s)\, ds\right\| < \infty\right\},$$

$$\mathscr{A}_I = \left\{A(t); \sup_{t\geqslant 0} \left\|\int_t^{t+1} A(s)\, ds\right\| < \infty\right\},$$

$$\mathscr{A}_C = \{A(t); A(t) \textit{ is continuous on } [0, \infty)\}.$$

We notice that

$$\mathscr{A}_A = \left\{A(t); \sup_{t\geqslant 0} e^{-t} \int_0^t e^s \|A(s)\|\, ds < \infty\right\}. \tag{2.8}$$

For the proof, see [7].

Let $p(t)$ be a continuous function on $[0, \infty)$ and let $P(t) = e^{-t} \int_0^t e^s p(s)\, ds$ and $Q(t) = \int_t^{t+1} p(s)\, ds$. Then it is easy to see that

$$P'(t) + P(t) = p(t), \tag{2.9}$$

$$Q(t) = P(t+1) - P(t) + \int_t^{t+1} P(s)\, ds. \tag{2.10}$$

For the above classes, we present the following lemma.

LEMMA 2.4.

$$\mathscr{A}_B \subsetneqq \mathscr{A}_E \subsetneqq \mathscr{A}_I \subsetneqq \mathscr{A}_C.$$

Proof. $\mathscr{A}_B \subsetneqq \mathscr{A}_A \subsetneqq \mathscr{A}_E$ and $\mathscr{A}_I \subsetneqq \mathscr{A}_C$ are trivial. We show that $\mathscr{A}_E \subsetneqq \mathscr{A}_I$. Let $A(t) \in \mathscr{A}_E$ and $\tilde{A}(t) = e^{-t} \int_0^t e^s A(s)\, ds$, then there exists $M > 0$ such that $0 \leqslant \|\tilde{A}(t)\| \leqslant M$ for all $t \geqslant 0$. Therefore by (2.10) we have

$$\left\|\int_t^{t+1} A(s)\, ds\right\| \leqslant \|\tilde{A}(t+1)\| + \|\tilde{A}(t)\| + \int_t^{t+1} \|\tilde{A}(s)\|\, ds \leqslant 3M$$

which implies $A(t) \in \mathscr{A}_I$. Let $A(t) = (t \sin 2\pi t) I$; then $A(t) \in \mathscr{A}_I$ but $\notin \mathscr{A}_E$. The proof is now complete.

For an $n \times m$ matrix $B = (b_{ij})$, let $\|B\| = \sum_{i=1}^n \sum_{j=1}^m |b_{ij}|$. The following lemma will help to simplify the proof of our main results.

LEMMA 2.5. *For $n \times m$ matrix valued continuous functions $B(t)$ on $[0, \infty)$,*

$$\sup_{t \geqslant 0} e^{-t} \int_0^t e^s \|B(s)\| \, ds < \infty$$

if and only if $$\sup_{t \geqslant 0} e^{-\mu t} \int_0^t e^{\mu s} \|B(s)\| \, ds < \infty \qquad \text{for any} \quad \mu > 0$$

and

$$\sup_{t \geqslant 0} \left\| e^{-t} \int_0^t e^s B(s) \, ds \right\| < \infty$$

if and only if $$\sup_{t \geqslant 0} \left\| e^{-\mu t} \int_0^t e^{\mu s} B(s) \, ds \right\| < \infty \qquad \textit{for any } \mu > 0.$$

The proof of Lemma 2.5 depends on variations of the relation (2.9) and we omit it.

Let Z denote any one of "UB," "UB and UltB," "UB and UUB." Define the perturbation class $\mathscr{G} = \mathscr{G}(\mathscr{A})$ for $\mathscr{A} \subset \mathscr{A}_C$ by

$$\mathscr{G}(\mathscr{A}) = \{g(t, y); \text{ for any } A(t) \in \mathscr{A} \text{ for which solutions of (L) are } Z, \text{ the solutions of (PL) are } Z\}.$$

Define also for $\mathscr{A} \subset \mathscr{A}_C$,

$$\mathscr{H}(\mathscr{A}) = \{h(t); \text{ for any } A(t) \in \mathscr{A} \text{ for which solutions of (L) are } Z, \text{ the solutions of } (\mathrm{PL}_h) \text{ are } Z\}.$$

We shall use Z for UB in Section 3, for UB and UltB in Section 4 and for UB and UUB in Section 5.

It is easy to show the following property of $\mathscr{G}$ and $\mathscr{H}$.

LEMMA 2.6. *Let $\mathscr{A}_\alpha \subset \mathscr{A}_\beta \subset \mathscr{A}_C$. Then*

$$\mathscr{G}(\mathscr{A}_\alpha) \supset \mathscr{G}(\mathscr{A}_\beta) \qquad \textit{and} \qquad \mathscr{H}(\mathscr{A}_\alpha) \supset \mathscr{H}(\mathscr{A}_\beta).$$

3. UNIFORM BOUNDEDNESS

In this section we shall determine the classes of perturbations which preserve uniform boundedness for the classes $\mathscr{A}_B$, $\mathscr{A}_A$, $\mathscr{A}_E$, $\mathscr{A}_I$ and $\mathscr{A}_C$.

Let G_0 be the class of functions $g(t, y)$ such that $\|g(t, y)\| \leqslant \gamma(t)\, \phi(\|y\|)$ for all $t \geqslant 0$ and $\|y\| \geqslant R$, where $\gamma(t)$ is a continuous function on $t \geqslant 0$ and

$\phi(r)$ is a positive, continuous and nondecreasing function on $r \geqslant r_0 > 0$ such that

$$\int_0^\infty \gamma(t)\,dt < \infty \qquad \text{and} \qquad \int_{r_0}^\infty \frac{dr}{\phi(r)} = \infty.$$

Let H and H_0 be the classes of functions given by

$$H = \left\{ h(t);\ \sup_{t \geqslant 0} \left\| \int_0^t h(s)\,ds \right\| < \infty \right\}$$

and

$$H_0 = \left\{ h(t);\ \int_0^\infty \|h(t)\|\,dt < \infty \right\}.$$

THEOREM 3.1. *For uniform boundedness and for the classes* $\mathscr{A}_B$, $\mathscr{A}_A$, $\mathscr{A}_E$, $\mathscr{A}_I$ *and* $\mathscr{A}_C$, *we have*

$$\mathscr{G}(\mathscr{A}_B) \supsetneqq \mathscr{G}(\mathscr{A}_C) \supset G_0,$$
$$H \supset \mathscr{H}(\mathscr{A}_B) \supset \mathscr{H}(\mathscr{A}_A) \supsetneqq \mathscr{H}(\mathscr{A}_E) = \mathscr{H}(\mathscr{A}_I) = \mathscr{H}(\mathscr{A}_C) = H_0.$$

Proof. By Lemma 2.6, $\mathscr{G}(\mathscr{A}_B) \supset \mathscr{G}(\mathscr{A}_C)$. We first prove that

$$\mathscr{G}(\mathscr{A}_C) \supset G_0. \tag{3.1}$$

Let $g(t, y) \in G_0, A(t) \in \mathscr{A}_C$ and suppose that the solutions of (L) are UB. Then it is well known [10, Theorem 19.1] that there exists a Liapunov function $V(t, x)$ defined on $t \geqslant 0$, $x \in \mathbf{R}^n$, satisfying the following conditions:

(i) $\|x\| \leqslant V(t, x) \leqslant k\,\|x\|$,

(ii) $|V(t, x) - V(t, y)| \leqslant k\|x - y\|$,

(iii) $\dot{V}_{(L)}(t, x) \leqslant 0$.

Then for $t \geqslant 0$ and $\|x\| \geqslant R$ we have

$$\begin{aligned} \dot{V}_{(\mathrm{PL})}(t, y) &\leqslant \dot{V}_{(\mathrm{L})}(t, y) + k\|g(t, y)\| \\ &\leqslant k\gamma(t)\,\phi(\|y\|) \\ &\leqslant k\gamma(t)\,\phi(V(t, y)). \end{aligned}$$

Then it follows from Lemma 2.1 that the solutions of (PL) are UB, which proves (3.1). It also shows that

$$\mathscr{H}(\mathscr{A}_C) \supset H_0. \tag{3.2}$$

We next prove that

$$H \supset \mathscr{H}(\mathscr{A}_B). \tag{3.3}$$

Suppose that there exists a sequence $\{t_m\}$ such that $t_m \to \infty$ and

$$\left\| \int_0^{t_m} h(s)\, ds \right\| \to \infty \qquad \text{as} \quad m \to \infty.$$

Let $A_0(t) \equiv 0$. Then $A_0(t) \in \mathscr{A}_B$ and the solutions of $x' = A_0(t)\, x$ are UB. The solution $y(t, t_0, 0)$ of $y' = A_0(t)\, y + h(t) = h(t)$ satisfies

$$\| y(t_m, t_0, 0)\| = \left\| \int_{t_0}^{t_m} h(s)\, ds \right\| \to \infty \qquad \text{as} \quad m \to \infty$$

which implies $h(t) \notin \mathscr{H}(\mathscr{A}_B)$. This proves (3.3).

By Lemmas 2.4 and 2.6, and (3.2) and (3.3), we have

$$H \supset \mathscr{H}(\mathscr{A}_B) \supset \mathscr{H}(\mathscr{A}_A) \supset \mathscr{H}(\mathscr{A}_E) \supset \mathscr{H}(\mathscr{A}_I) \supset \mathscr{H}(\mathscr{A}_C) \supset H_0. \tag{3.4}$$

We next prove that

$$\mathscr{H}(\mathscr{A}_E) \subset H_0. \tag{3.5}$$

Suppose that $h(t) \in \mathscr{H}(\mathscr{A}_E)$ and

$$\int_0^\infty \|h(t)\|\, dt = \infty. \tag{3.6}$$

Then

$$\int_0^\infty |h_i(t)|\, dt = \infty \tag{3.7}$$

for some component h_i of h. Let I^-, I^+ and I^0 be the sets of points $t \geqslant 0$, given by

$$\begin{aligned} I^- &= \{t \geqslant 0; h_i(t) \leqslant -(1+t^2)^{-1}\}, \\ I^+ &= \{t \geqslant 0; h_i(t) \geqslant (1+t^2)^{-1}\}, \\ I^0 &= \{t \geqslant 0; |h_i(t)| < (1+t^2)^{-1}\}. \end{aligned}$$

We can define a C^1-function $a(t)$ on $[0, \infty)$, given by Strauss and Yorke [9], such that $0 \leqslant a(t) \leqslant \log 3$ for $t \geqslant 0$, and

$$\begin{aligned} a(t) &= 0 && \text{if} \quad t \in I^-, \\ &= \log 3 && \text{if} \quad t \in I^+. \end{aligned}$$

Then we obtain for all $t \geqslant 0$,

$$h_i(t)(e^{a(t)} - 2) + (1 + t^2)^{-1} \geqslant |h_i(t)| - (1 + t^2)^{-1}.$$

Hence

$$\int_0^t h_i(s)(e^{a(s)} - 2)\, ds \geqslant \int_0^t |h_i(s)|\, ds - \int_0^t \frac{2}{1 + s^2}\, ds.$$

Note that $h_i(t) \in \mathscr{H}(\mathscr{A}_E) \subset H$. Then, by (3.7) we have

$$\int_0^\infty h_i(t)\, e^{a(t)}\, dt = \infty. \tag{3.8}$$

Consider the systems

$$x' = -a'(t)\, Ix, \tag{3.9}$$

$$y' = -a'(t)\, Iy + h(t). \tag{3.10}$$

Let $A(t) = -a'(t)\, I$. We have

$$\left\| e^{-t} \int_0^t e^s A(s)\, ds \right\| \leqslant 3\, \|I\| \log 3 \qquad \text{for all} \quad t \geqslant 0,$$

hence $A(t) \in \mathscr{A}_E$. A fundamental matrix for (3.9) is given by $X(t) = e^{-a(t)} I$. Since

$$\|X(t)\, X^{-1}(s)\| = \|e^{-a(t)} e^{a(s)} I\| \leqslant 3 \|I\| \qquad \text{for} \quad t \geqslant s \geqslant 0,$$

the solutions of (3.9) are UB. But if y_i denotes the ith component of the solution of (3.10), then by (3.8),

$$\begin{aligned} y_i(t, t_0, 0) &= e^{-a}(t) \int_{t_0}^t e^{a(s)} h_i(s)\, ds \\ &\geqslant 3^{-1} \int_{t_0}^t h_i(s)\, e^{a(s)}\, ds \to \infty \qquad \text{as} \quad t \to \infty \end{aligned}$$

for each $t_0 \geqslant 0$, which implies the solutions of (3.10) are not UB. But it is impossible, since $h \in \mathscr{H}(\mathscr{A}_E)$. This contradicts (3.6). Therefore, h satisfies $\int_0^\infty \|h(t)\|\, dt < \infty$, which proves (3.5).

Finally we show that there exists $h(t)$ such that $h \notin \mathscr{H}(\mathscr{A}_E) = H_0$ and

$h \in \mathscr{H}(\mathscr{A}_A)$. If $h^*(t) = -\int_t^\infty h(s)\,ds$ is defined on $[0, \infty)$, then by an integration by parts, the solutions of (PL_h) satisfy

$$y(t, t_0, y_0) = X(t)\,X^{-1}(t_0)\,y_0 + h^*(t) - X(t)\,X^{-1}(t_0)\,h^*(t_0) + X(t)\int_{t_0}^t X^{-1}(s)\,A(s)\,h^*(s)\,ds. \tag{3.11}$$

Let $h(t) = (\cos e^t, 0,..., 0)$. Then $\int_0^\infty \|h(t)\|\,dt = \infty$. Sine $\|\int_s^t h(u)\,du\| \leqslant 2e^{-s}$ for all $t \geqslant s \geqslant 0$, $h^*(t) \equiv -\int_t^\infty h(s)\,ds$ is defined and $\|h^*(t)\| \leqslant 2e^{-t}$ for all $t \geqslant 0$. Suppose that $A(t) \in \mathscr{A}_A$ (i.e., there exists $M > 0$ such that $\int_t^{t+1} \|A(s)\|\,ds \leqslant M$ for all $t \geqslant 0$) and solutions of (L) are UB. It follows then from (3.11) and Lemma 2.3 that

$$\begin{aligned}\|y(t, t_0, y_0)\| &\leqslant K\|y_0\| + 2e^{-t} + 2Ke^{-t_0} + 2K\int_{t_0}^t e^{-s}\|A(s)\|\,ds \\ &\leqslant K\|y_0\| + 2(K+1) + 2K(1-e^{-1})^{-1}M,\end{aligned}$$

which implies that the solutions of (PL) are UB. Thus we have $h(t) \in \mathscr{H}(\mathscr{A}_A)$ and $h(t) \notin \mathscr{H}(\mathscr{A}_E) = H_0$. The proof of Theorem 3.1 is now complete.

4. Uniform Boundedness and Ultimate Boundedness

In this section we shall determine the classes of perturbations which preserve uniform boundedness and ultimate boundedness for the classes $\mathscr{A}_B$, $\mathscr{A}_A$, $\mathscr{A}_E$, $\mathscr{A}_I$ and $\mathscr{A}_C$. The classes of functions $\mathscr{G}(\mathscr{A})$ and $\mathscr{H}(\mathscr{A})$ in question here are those concerned with "UB and UltB." To make explicit the difference with the statement in Section 3, we denote these perturbation classes here by $\mathscr{G}^*(\mathscr{A}_B)$, $\mathscr{G}^*(\mathscr{A}_C)$, $\mathscr{H}^*(\mathscr{A}_B)$, $\mathscr{H}^*(\mathscr{A}_A)$, $\mathscr{H}^*(\mathscr{A}_E)$, $\mathscr{H}^*(\mathscr{A}_I)$ and $\mathscr{H}^*(\mathscr{A}_C)$.

Theorem 4.1. *For uniform boundedness and ultimate boundedness and for the classes $\mathscr{A}_B$, $\mathscr{A}_A$, $\mathscr{A}_E$, $\mathscr{A}_I$* and $\mathscr{A}_C$, we have

$$\mathscr{G}^*(\mathscr{A}_B) \supsetneq \mathscr{G}^*(\mathscr{A}_C) \supset G_0,$$

$$H \supset \mathscr{H}^*(\mathscr{A}_B) \supset \mathscr{H}^*(\mathscr{A}_A) \supsetneq \mathscr{H}^*(\mathscr{A}_E) = \mathscr{H}^*(\mathscr{A}_I) = \mathscr{H}^*(\mathscr{A}_C) = H_0.$$

Proof. By Lemma 2.6, $\mathscr{G}^*(\mathscr{A}_B) \supset \mathscr{G}^*(\mathscr{A}_C)$. We first prove that

$$\mathscr{G}^*(\mathscr{A}_C) \supset G_0. \tag{4.1}$$

Let $A(t) \in \mathscr{A}_C$ and suppose that the solutions of (L) be UB and UltB. Let $g \in G_0$. We must show that the solutions of (PL) are UB and UltB.

Let $X(t)$ be a fundamental matrix for (L). Then by Lemma 2.3 there exists $K \geqslant 1$ such that

$$\|X(t)\, X^{-1}(s)\| \leqslant K \qquad \text{for all} \quad t \geqslant s \geqslant 0$$

and (4.2)

$$\|X(t)\| \to 0 \qquad \text{as} \quad t \to \infty.$$

It follows from Theorem 3.1 that the solutions of (PL) are UB; i.e., for any $\alpha > 0$, there exists $\beta(\alpha) > 0$ such that $\| y_0 \| \leqslant \alpha$ implies

$$\| y(t, t_0, y_0)\| < \beta(\alpha) \qquad \text{for all} \quad t_0 \geqslant 0 \qquad \text{and} \quad t \geqslant t_0 \geqslant 0.$$

Let $B = \beta(R)$. Then the existence of $t_1 \geqslant t_0$ such that $\| y(t_1, t_0, y_0)\| \leqslant R$ implies

$$\| y(t, t_0, y_0)\| < B \qquad \text{for all} \quad t \geqslant t_1 .$$

Hence for the proof of ultimate boundedness of solutions of (PL), it suffices to show that for $t_0 \geqslant 0$ and $\| y_0 \| > B$, the solution $y(t, t_0, y_0)$ of (PL) satisfies $\| y(t_1, t_0, y_0)\| \leqslant R$ at some $t_1 \geqslant t_0$. Assume that $\| y(t, t_0, y_0)\| > R$ for all $t \geqslant t_0$. By the fact that $\int_0^\infty \gamma(s)\, ds < \infty$ and by (4.2), we can choose $t_2 > t_0$ so large that

$$K\phi(\beta(\| y_0 \|)) \int_{t_2}^\infty \gamma(s)\, ds \leqslant R/3$$

and

$$\|X(t)\| \, \|X^{-1}(t_0)\| \, \| y_0 \| \leqslant R/3 \qquad \text{for} \quad t \geqslant t_2 .$$

Then it follows from (4.2) that for $t \geqslant t_0$,

$$\begin{aligned}
\| y(t)\| &= \| y(t, t_0, y_0)\| \\
&\leqslant \|X(t)\| \, \|X^{-1}(t_0)\| \, \| y_0 \| + \|X(t)\| \int_{t_0}^{t_2} \|X^{-1}(s)\ g(s, y(s))\| \, ds \\
&\quad + \int_{t_2}^{t} \|X(t)\, X^{-1}(s)\| \, \gamma(s)\, \phi(\| y(s)\|)\, ds \\
&\leqslant \|X(t)\| \, \|X^{-1}(t_0)\| \, \| y_0 \| \\
&\quad + \|X(t)\| (t_2 - t_0) \sup_{t_0 \leqslant s \leqslant t_2} \|X^{-1}(s)\ g(s, y(s))\| \\
&\quad + K\phi(\beta(\| y_0 \|)) \int_{t_2}^{t} \gamma(s)\, ds,
\end{aligned}$$

because $\phi(r)$ is a nondecreasing function. Choose $t_1 \geqslant t_2$ so large that

$$\|X(t)\|(t_2 - t_0) \sup_{t_0 \leqslant s \leqslant t_2} \|X^{-1}(s)\, g(s, y(s))\| \leqslant R/3 \qquad \text{for} \quad t \geqslant t_1.$$

Then, we have $\| y(t_1)\| \leqslant R$. But this contradicts the assumption. Therefore, every solution $y(t, t_0, x_0)$ satisfies $\| y(t, t_0, y_0)\| < B$ for $t \geqslant t_1$, which proves (4.1). This also proves that

$$\mathscr{H}^*(\mathscr{A}_C) \supset H_0. \tag{4.3}$$

We next prove that

$$H \supset \mathscr{H}^*(\mathscr{A}_B). \tag{4.4}$$

Suppose that there exists a sequence $\{t_k\}$ such that $t_k \to \infty$ and

$$\left\| \int_0^{t_k} h(s)\, ds \right\| \to \infty \qquad \text{as} \quad k \to \infty.$$

We may assume without loss of generality that there exist increasing subsequences $\{u_m\} \subset \{t_k\}$ and $\{v_m\} \subset \{t_k\}$ such that $u_m < v_m < u_{m+1}$, $u_m \to \infty, v_m \to \infty$ as $m \to \infty$ and

$$\left\| \int_{u_m}^{v_m} h(s)\, ds \right\| \geqslant m.$$

We can define a C^1-function $p(t)$ on $[0, \infty)$ such that $p(t) \equiv 1$ for $0 \leqslant t \leqslant u_1$, $p(t) \equiv 1/m$ for $u_m \leqslant t \leqslant v_m$, $| p'(t)| \leqslant (m+1)^{-1}$ for $v_m \leqslant t \leqslant u_{m+1}$, and p is monotone on $[v_m, u_{m+1}]$ for every positive integer m. Consider the systems

$$x' = (p'(t)/p(t))\, Ix \tag{4.5}$$

and

$$y' = (p'(t)/p(t))\, Iy + h(t). \tag{4.6}$$

Let $A(t) = (p'(t)/p(t))\, I$. Then $A(t) \in \mathscr{A}_B$, since $| p'(t)/p(t)| \leqslant 1$ for all $t \geqslant 0$. A fundamental matrix for (4.5) is given by $X(t) = p(t)\, I$. Then the solutions of (4.5) are UB and UltB, since

$$\|X(t)\, X^{-1}(s)\| = \|(p(t)/p(s))\, I\| \leqslant \|I\| \qquad \text{for all} \quad t \geqslant s \geqslant 0$$

and $p(t) \to 0$ as $t \to \infty$. The solution of (4.6) satisfies

$$\| y(v_m, u_m, 0)\| = \left\| p(v_m) \int_{u_m}^{v_m} p(s)^{-1} Ih(s)\, ds \right\|$$
$$= \left\| m^{-1} \int_{u_m}^{v_m} mh(s)\, ds \right\| \geqslant m$$

for every m, which implies that solutions of (4.6) are not UB. Thus $h(t) \notin \mathscr{H}^*(\mathscr{A}_B)$, which proves (4.4).

By Lemmas 2.4 and 2.6, and (4.3) and (4.4), we have

$$H \supset \mathscr{H}^*(\mathscr{A}_B) \supset \mathscr{H}^*(\mathscr{A}_A) \supset \mathscr{H}^*(\mathscr{A}_E) \supset \mathscr{H}^*(\mathscr{A}_I) \supset \mathscr{H}^*(\mathscr{A}_C) \supset H_0. \quad (4.7)$$

We next prove that

$$\mathscr{H}^*(\mathscr{A}_E) \subset H_0. \quad (4.8)$$

Let $h(t) \in \mathscr{H}^*(\mathscr{A}_E)$. Assume that $\int_0^\infty \|h(t)\|\, dt = \infty$. The proof of (3.5) is imitated line by line through the proof of (3.8). Then we can choose an increasing sequence $\{t_m\}$ such that $t_m \to \infty$ as $m \to \infty$, $t_{m+1} > t_m + 1$ and

$$\int_{t_m}^{t_{m+1}} h_i(s)\, e^{a(s)}\, ds > m.$$

Define a continuous function ψ: $[0, \infty) \to [-1, 0]$ such that $\psi(t) \equiv 0$ for $t_{2m} \leqslant t \leqslant t_{2m+1}$ and

$$\int_{t_{2m-1}}^{t_{2m}} \psi(s)\, ds = -1.$$

Consider the systems

$$x' = (\psi(t) - a'(t))\, Ix \quad (4.9)$$

and

$$y' = (\psi(t) - a'(t))\, Iy + h(t). \quad (4.10)$$

Let $A(t) = (\psi(t) - a'(t))\, I$; then we have

$$\left\| e^{-t} \int_0^t e^s A(s)\, ds \right\| \leqslant (1 + 3 \log 3)\|I\| \qquad \text{for all} \quad t \geqslant 0.$$

Hence $A(t) \in \mathscr{A}_E$. A fundamental matrix for (4.9) is given by

$$X(t) = I \exp\left(\int_0^t \psi(u)\, du - a(t)\right).$$

Then the solutions of (4.9) are UB, since

$$\|X(t)\,X^{-1}(s)\| = \left\|I \exp\left(\int_s^t \psi(u)\,du - a(t) + a(s)\right)\right\|$$

$$\leqslant 3\|I\| \qquad \text{for all} \quad t \geqslant s \geqslant 0.$$

Since

$$\int_0^t \psi(u)\,du \to -\infty \qquad \text{as} \quad t \to \infty,$$

solutions of (4.9) are UltB. The ith component of the solution of (4.10) satisfies

$$y_i(t_{2m+1}, t_{2m}, 0) = \int_{t_{2m}}^{t_{2m+1}} h_i(s) \exp[a(s) - a(t_{2m+1})]\,ds$$

$$\geqslant 3^{-1} \int_{t_{2m}}^{t_{2m+1}} h_i(s)\, e^{a(s)}\,ds > \tfrac{2}{3} m$$

for every positive integer m, which implies that the solutions of (4.10) are not UB. But it is impossible, since $h \in \mathscr{H}^*(\mathscr{A}_E)$. This contradicts the assumption that $\int_0^\infty \|h(t)\|\,dt = \infty$. Therefore, $\int_0^\infty \|h(t)\|\,dt < \infty$, which proves (4.8).

Finally we exhibit an example $h(t)$ such that $h(t) \in \mathscr{H}^*(\mathscr{A}_A)$ and $h(t) \notin H_0$. Let $h(t) = (\sin e^t, 0,\dots, 0)$. Then $\int_0^\infty \|h(t)\|\,dt = \infty$, $h^*(t) = -\int_t^\infty h(s)\,ds$ is defined and $\|h^*(t)\| \leqslant 2e^{-t}$ for all $t \geqslant 0$. Suppose that there exists $M > 0$ such that $\int_t^{t+1} \|A(s)\|\,ds \leqslant M$ for all $t \geqslant 0$ and solutions of (L) are UB and UltB. It follows then from (3.11) and Lemma 2.3 that

$$\|y(t, t_0, y_0)\| \leqslant K\|y_0\| + 2(K+1) + 2K(1-e^{-1})^{-1} M$$

and

$$\|y(t, t_0, y_0)\| \leqslant \|X(t)\|\,\|X^{-1}(t_0)\,y_0\| + 2(K+1) + 2K(1-e^{-1})^{-1} M,$$

which implies that the solutions of (PL) are UB and UltB. Hence we have $h(t) \in \mathscr{H}^*(\mathscr{A}_A)$. The proof of Theorem 4.1 is now complete.

5. Uniform Boundedness and Uniform Ultimate Boundedness

In this section we shall determine the classes which preserve uniform boundedness and uniform ultimate boundedness for the classes $\mathscr{A}_B$, $\mathscr{A}_A$, $\mathscr{A}_E$, $\mathscr{A}_I$ and $\mathscr{A}_C$. The classes of functions $\mathscr{G}(\mathscr{A})$ and $\mathscr{H}(\mathscr{A})$ in question here are those concerned with "UB and UUB." To make explicit the difference with the statements in Sections 3 and 4, we denote these perturbation classes here by $\tilde{\mathscr{G}}(\mathscr{A}_B)$, $\tilde{\mathscr{G}}(\mathscr{A}_C)$, $\tilde{\mathscr{H}}(\mathscr{A}_B)$, $\tilde{\mathscr{H}}(\mathscr{A}_A)$, $\tilde{\mathscr{H}}(\mathscr{A}_E)$, $\tilde{\mathscr{H}}(\mathscr{A}_I)$ and $\tilde{\mathscr{H}}(\mathscr{A}_C)$.

Let $\tilde{G}_0$, $\tilde{H}$, and $\tilde{H}_0$ be the classes of functions given by

$$\tilde{G}_0 = \Big\{ g(t, y); \|g(t, y)\| \leqslant \gamma(t)\|y\|^{\beta} \qquad (0 \leqslant \beta < 1) \text{ for all } t \geqslant 0$$

$$\text{and } \|y\| \geqslant R, \text{ where } \sup_{t \geqslant 0} e^{-t} \int_0^t e^s \gamma(s)\, ds < \infty \Big\},$$

$$\tilde{H} = \Big\{ h(t); \sup_{t \geqslant 0} \Big\| e^{-t} \int_0^t e^s h(s)\, ds \Big\| < +\infty \Big\}$$

and

$$\tilde{H}_0 = \Big\{ h(t); \sup_{t \geqslant 0} e^{-t} \int_0^t e^s \|h(s)\|\, ds < +\infty \Big\}.$$

THEOREM 5.1. *For the uniform boundedness and uniform ultimate boundedness and for the classes $\mathscr{A}_B$, $\mathscr{A}_A$, $\mathscr{A}_E$, $\mathscr{A}_I$ and $\mathscr{A}_C$, we have*

$$\tilde{\mathscr{G}}(\mathscr{A}_B) \supsetneqq \tilde{\mathscr{G}}(\mathscr{A}_C) \supset \tilde{G}_0,$$
$$\tilde{H} = \tilde{\mathscr{H}}(\mathscr{A}_B) = \tilde{\mathscr{H}}(\mathscr{A}_A) \supsetneqq \tilde{\mathscr{H}}(\mathscr{A}_E) = \tilde{\mathscr{H}}(\mathscr{A}_I) = \tilde{\mathscr{H}}(\mathscr{A}_C) = \tilde{H}_0.$$

THEOREM 5.2. *Suppose that $\sup_{t \geqslant 0} e^{-t} \int_0^t e^s \|A(s)\|\, ds < +\infty$ and the solutions of (L) are UB and UUB. Then the solutions of (PL_h) are UB and UUB if and only if $h(t)$ satisfies*

$$\sup_{t \geqslant 0} \Big\| e^{-t} \int_0^t e^s h(s)\, ds \Big\| < +\infty.$$

Proof of Theorem 5.1. By Lemma 2.6, $\tilde{\mathscr{G}}(\mathscr{A}_B) \supset \tilde{\mathscr{G}}(\mathscr{A}_C)$. We first prove that

$$\tilde{\mathscr{G}}(\mathscr{A}_C) \supset \tilde{G}_0. \tag{5.1}$$

Let $g(t, y) \in \tilde{G}_0$, $A(t) \in \mathscr{A}_C$ and suppose that the solutions of (L) are UB and UUB. Then it is well known [10, Theorem 19.1] that there exists a Liapunov function $V(t, x)$ defined on $t \geqslant 0$, $x \in \mathbf{R}^n$, satisfying the following conditions;

(i) $\|x\| \leqslant V(t, x) \leqslant k\|x\|$,

(ii) $|V(t, x) - V(t, y)| \leqslant k\|x - y\|$,

(iii) $\dot{V}_{(\mathrm{L})}(t, x) \leqslant -cV(t, x)$ for some $c > 0$.

Hence for $t \geqslant 0$ and $\| y\| \geqslant R$ we have

$$\begin{aligned}\dot{V}_{(\mathrm{PL})}(t, y) &\leqslant \dot{V}_{(\mathrm{L})}(t, y) + k\| g(t, y)\| \\ &\leqslant -cV(t, y) + k\gamma(t)\{V(t, y)\}^{\beta}.\end{aligned} \tag{5.2}$$

It follows from Lemma 2.2 that the solutions of (PL) are UB and UUB, which proves (5.1). This also shows that

$$\tilde{\mathscr{H}}(\mathscr{A}_C) \supset \tilde{H}_0. \tag{5.3}$$

We next show that

$$\tilde{H} \supset \tilde{\mathscr{H}}(\mathscr{A}_B). \tag{5.4}$$

Suppose that there exists a sequence $\{t_m\}$ such that $t_m \to \infty$ and

$$\left\| e^{-t_m} \int_0^{t_m} e^s h(s)\, ds \right\| \to \infty \qquad \text{as} \quad m \to \infty.$$

We consider the systems

$$x' = -Ix, \tag{5.5}$$

$$y' = -Iy + h(t). \tag{5.6}$$

Let $A(t) = -I$; then $A(t) \in \mathscr{A}_B$. A fundamental matrix for (5.5) is given by $X(t) = e^{-t}I$. Since

$$\|X(t)\, X^{-1}(s)\| = \|I\|\, e^{-(t-s)} \qquad \text{for all} \quad t \geqslant s \geqslant 0,$$

the solutions of (5.5) are UB and UUB. But the solution of (5.6) satisfies

$$\| y(t_m, 0, 0)\| = \left\| e^{-t_m} \int_0^{t_m} e^s h(s)\, ds \right\| \to \infty \qquad \text{as} \quad m \to \infty,$$

which implies $h(t) \notin \tilde{\mathscr{H}}(\mathscr{A}_B)$. This proves (5.4).

By Lemmas 2.4 and Lemma 2.6, and (5.3) and (5.4) we have

$$\tilde{H} \supset \tilde{\mathscr{H}}(\mathscr{A}_B) \supset \tilde{\mathscr{H}}(\mathscr{A}_A) \supset \tilde{\mathscr{H}}(\mathscr{A}_E) \supset \tilde{\mathscr{H}}(\mathscr{A}_I) \supset \tilde{\mathscr{H}}(\mathscr{A}_C) \supset \tilde{H}_0 \tag{5.7}$$

and

$$\tilde{H} \supsetneqq \tilde{H}_0. \tag{5.8}$$

We next prove that

$$\tilde{H} \subset \tilde{\mathscr{H}}(\mathscr{A}_A). \tag{5.9}$$

Let $A(t) \in \mathscr{A}_A$ and suppose that the solutions of (L) are UB and UUB. By (2.8) and Lemma 2.5, there exists $L > 0$ such that $e^{-\lambda t} \int_0^t e^{\lambda s} \|A(s) + I\| \, ds \leqslant L$ for λ in Lemma 2.3 and $t \geqslant 0$. Suppose that there exists $M > 0$ such that $\|e^{-t} \int_0^t e^s h(s) \, ds\| \leqslant M$ for $t \geqslant 0$. Let $\tilde{h}(t) = e^{-t} \int_0^t e^s h(s) \, ds$. Then by (2.9) and an integration by parts, the solution of (PL_h) satisfies

$$y(t, t_0, y_0) = X(t) X^{-1}(t_0) \, y_0 + \tilde{h}(t) - X(t) X^{-1}(t_0) \, \tilde{h}(t_0) + X(t) \int_{t_0}^t X^{-1}(s)(A(s) + I) \, \tilde{h}(s) \, ds.$$

Then

$$\begin{aligned} \| y(t, t_0, y_0)\| &\leqslant K \| y_0\| \, e^{-\lambda(t - t_0)} + M + KMe^{-\lambda(t - t_0)} \\ &\quad + KMe^{-\lambda t} \int_{t_0}^t e^{\lambda s} \, \|A(s) + I\| \, ds \\ &\leqslant K \| y_0\| \, e^{-\lambda(t - t_0)} + M + KM + KLM \end{aligned}$$

which implies the solutions of (PL_h) are UB and UUB, proving (5.9).

Finally we show that

$$\tilde{\mathscr{H}}(\mathscr{A}_E) \subset \tilde{H}_0. \tag{5.10}$$

Let $h(t) \in \tilde{\mathscr{H}}(\mathscr{A}_E)$ and $\{t_m\}$ be a sequence such that $t_m \to \infty$ and

$$e^{-t_m} \int_0^{t_m} e^s \|h(s)\| \, ds \to \infty \qquad \text{as} \quad m \to \infty. \tag{5.11}$$

Then $e^{-t_m} \int_0^{t_m} e^s |h_i(s)| \, ds \to \infty$ as $m \to \infty$ for some component h_i of h. We construct the same C^1-function $a(t)$ as in the proof of Theorem 3.1. Then we obtain, for all $t \geqslant 0$,

$$h_i(t)(e^{a(t)} - 2) + (1 + t^2)^{-1} \geqslant |h_i(t)| - (1 + t^2)^{-1}.$$

Hence

$$\begin{aligned} &e^{-t_m} \int_0^{t_m} e^s h_i(s)(e^{a(s)} - 2) \, ds \\ &\quad \geqslant e^{-t_m} \int_0^{t_m} e^s |h_i(s)| \, ds - 2e^{-t_m} \int_0^{t_m} e^s (1 + s^2)^{-1} \, ds \\ &\quad \geqslant e^{-t_m} \int_0^{t_m} e^s |h_i(s)| \, ds - \pi. \end{aligned}$$

Therefore, by (5.7) and (5.11), we have

$$e^{-t_m}\int_0^{t_m} e^{s+a(s)}h_i(s)\,ds \to \infty \qquad \text{as} \quad m\to\infty. \tag{5.12}$$

Consider the systems

$$x' = -(1+a'(t))\,Ix, \tag{5.13}$$

$$y' = -(1+a'(t))\,Iy + h(t). \tag{5.14}$$

Let $A(t) = -(1+a'(t))\,I$. We have

$$\left\| e^{-t}\int_0^t e^s A(s)\,ds \right\| \leqslant (1+3\log 3)\|I\| \qquad \text{for all} \quad t\geqslant 0,$$

hence $A(t)\in \mathscr{A}_E$. A fundamental matrix for (5.13) is given by $X(t) = e^{-t-a(t)}I$. Since

$$\|X(t)\,X^{-1}(s)\| \leqslant 3\,\|I\|\,e^{-(t-s)} \qquad \text{for all} \quad t\geqslant s\geqslant 0,$$

the solutions of (5.13) are UB and UUB. But, if y_i denotes the ith component of the solution of (5.14), then by (5.12),

$$\begin{aligned} y_i(t_m, 0, 0) &= e^{-t_m-a(t_m)}\int_0^{t_m} e^{s+a(s)}\,h_i(s)\,ds \\ &\geqslant \tfrac{1}{3}\,e^{-t_m}\int_0^{t_m} e^{s+a(s)}h_i(s)\,ds \to \infty \qquad \text{as} \quad m\to\infty, \end{aligned}$$

which implies the solutions of (5.14) are not UB. But it is impossible since $h\in \mathscr{H}(\mathscr{A}_E)$. This contradicts (5.11). Therefore, h satisfies $\sup_{t\geqslant 0} e^{-t}\int_0^t e^s\|h(s)\|\,ds < \infty$, which proves (5.10). The proof of Theorem 5.1 is now complete.

Proof of Theorem 5.2. The sufficiency is immediate from Theorem 5.1. So we prove the necessity. Suppose that for $A(t)\in\mathscr{A}_A$ the solutions of (PL) are UB. Then there exists $L>0$ such that $e^{-t}\int_0^t e^s\|A(s)+I\|\,ds\leqslant L$ for all $t\geqslant 0$. Let $y(t) = y(t,0,0)$ be a solution of (PL_h). Then there exists $M>0$ such that $\|y(t)\|\leqslant M$ for $t\geqslant 0$. Notice that $h(t) = y'(t) - A(t)\,y(t)$. Then we have

$$\begin{aligned} \int_0^t e^s h(s)\,ds &= \int_0^t e^s y'(s)\,ds - \int_0^t e^s A(s)\,y(s)\,ds \\ &= e^t y(t) - \int_0^t e^s(A(s)+I)\,y(s)\,ds. \end{aligned}$$

Hence, $h(t)$ satisfies

$$\left\| e^{-t} \int_0^t e^s h(s)\, ds \right\| \leqslant (1+L) M \qquad \text{for all} \quad t \geqslant 0.$$

The proof of Theorem 5.2 is now complete.

Remarks to Theorem 5.1. (i) In Theorem 5.1, we have assumed $0 \leqslant \beta < 1$. It would be of interest to see whether Theorem 5.1 (i.e., $\mathscr{G}(\mathscr{A}_B) \supsetneqq \mathscr{G}(\mathscr{A}_C) \supset \tilde{G}_0$) holds if $\beta = 1$. The following example shows that we cannot make $\beta = 1$. Consider a linear equation

$$x' = -x. \tag{5.15}$$

The solutions of (5.15) are of course UB and UUB. Consider then,

$$y' = -y + 2y. \tag{5.16}$$

We see that all the solutions (except for zero) of (5.16) are unbounded.

(ii) Lemma 2.2 and (5.2) show that we can extent the class $\tilde{G}_0$ as follows.

$$\| g(t, y) \| \leqslant \gamma_1(t) \| y \| + \gamma_2(t) \| y \|^{\beta} \qquad (0 \leqslant \beta < 1)$$

for all $t \geqslant 0$ and $\| y \| \geqslant R$,

where $\displaystyle \limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v} \int_t^{t+v} \gamma_1(s)\, ds$ is sufficiently small

and $\displaystyle \sup_{t \geqslant 0} e^{-t} \int_0^t e^s \gamma_2(s)\, ds < \infty.$

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